THE LATTICE GLUON PROPAGATOR INTO THE NEXT MILLENNIUM

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We evaluate numerically the momentum-space gluon propagator in the lattice Landau gauge, for three- and four-dimensional pure $SU(2)$ lattice gauge theory. Although there are large finite-size effects, we always observe, in the limit of large lattice volumes, a gluon propagator decreasing in the infrared limit. This result can be interpreted in a straightforward way, by considering the proximity of the so-called first Gribov horizon in the infrared directions. We also consider the problem of discretization errors introduced by the lattice regularization, and their effect on the ultraviolet behavior of the gluon propagator.

1 Introduction

The gluon propagator is not an observable, since it is a gauge-dependent quantity. Nevertheless, the study of its infrared behavior provides us with a powerful tool for increasing our understanding of confinement in non-Abelian gauge theories. [1] In fact, the infrared behavior of the gluon propagator can be directly related to the behavior of the Wilson loop at large separations and to the existence of an area law. [2]

The theoretical predictions for the behavior of the Landau-gauge gluon propagator in the infrared limit – obtained mostly by solving the gluon Dyson-Schwinger equation *approximately* – range from a p^{-4} singularity [3] to a vanishing propagator. [4] A true nonperturbative investigation of these predictions is possible by performing Monte Carlo simulations of QCD on the lattice.

2 The lattice setup

Let us consider a standard Wilson action for $SU(2)$ lattice gauge theory in d dimensions. The lattice gluon propagator (in momentum space) can be written as

$$D(0) \equiv \frac{1}{d} \sum_{\mu=1}^{d} D_{\mu}(0) \tag{1}$$

$$D(k) \equiv \frac{1}{d-1} \sum_{\mu=1}^{d} D_{\mu}(k) \; , \tag{2}$$

where

$$D_\mu(k) \;\equiv\; \frac{\mathrm{Tr}}{6\,V}\,\langle\, \widetilde{A}_\mu(k)\, \widetilde{A}_\mu(-k)\,\rangle\;. \tag{3}$$

Here V is the lattice volume,

$$\widetilde{A}_\mu(k) \;\equiv\; \sum_x A_\mu(x)\, \exp\left[2\pi i\, (k\cdot x + k_\mu/2)\right] \tag{4}$$

and the lattice gluon field $A_\mu(x)$ is given by

$$A_\mu(x) \;\equiv\; \frac{1}{2\,i}\,\left[\, U_\mu(x) \;-\; U^\dagger_\mu(x)\,\right]\;. \tag{5}$$

In order to fix the lattice Landau gauge we can look for a local minimum of the functional

$$\mathcal{E}_U[g] \;\equiv\; 1 \;-\; \frac{\mathrm{Tr}}{2\,d\,V}\sum_{\mu=1}^{d}\,\sum_x\,\left[\, g(x)\, U_\mu(x)\, g^\dagger(x+e_\mu)\,\right]\;. \tag{6}$$

In fact, if the configuration $\{U_\mu\,(x)\}$ is a *stationary point* of the functional $\mathcal{E}_U[g]$ then the lattice divergence of $A_\mu(x)$ is zero, i.e.,

$$\left(\nabla\cdot A\right)(x) \equiv \sum_{\mu=1}^{d}\,\left[A_\mu(x) - A_\mu(x-e_\mu)\right] \;=\; 0\;. \tag{7}$$

This is the lattice formulation of the usual (continuum) Landau gauge-fixing condition. Moreover, requiring this stationary point to be a *minimum* of the functional $\mathcal{E}_U[g]$ implies that the transverse gauge-fixed configurations belong to the region Ω delimited by the so-called first Gribov horizon, defined as the set of configurations for which the smallest non-trivial eigenvalue of the Faddeev-Popov operator is zero.

Thus, when the lattice Landau gauge is imposed, the physical configuration space is restricted to the region Ω and one can prove [5] a *rigorous* inequality for the Fourier components of the gluon field $A_\mu(x)$. From this inequality it follows that the region Ω is bounded by a certain ellipsoid Θ. This bound implies the proximity of the first Gribov horizon in infrared directions and the consequent suppression of the low-momentum components of the gauge field. This bound also causes a strong suppression of the gluon propagator in the infrared limit. In fact, Zwanziger proved [6] that, in four dimensions and in the infinite-volume limit, the gluon propagator is less singular than p^{-2} in the infrared limit and that, very likely, it *does* vanish in this limit. A similar

result holds in three dimensions: one obtains that, in the infinite-volume limit, the gluon propagator must be less singular than p^{-1} as $p \to 0$ and that, very likely, it vanishes in the infrared limit. We remark that these predictions for the gluon propagator are *β-independent*: in fact, they are derived only from the positiveness of the Faddeev-Popov operator when the lattice Landau gauge is imposed.

3 The infrared behavior of the gluon propagator

We have studied the momentum-space gluon propagator $D(k)$ in four [7] and in three [8] dimensions. In both cases we have found that, if the lattice volume V is large enough, the gluon propagator is decreasing as the magnitude of the lattice momentum $p(k) \equiv 2 \left[\sum_{\mu=1}^{d} \sin^2 (\pi \, k_\mu) \right]^{1/2}$ decreases, provided that $p(k)$ is smaller than a value p_{dec}. Also, the lattice volume at which this behavior for the gluon propagator starts to be observed increases with the coupling β, i.e., finite-size effects are very large in the small-momenta sector. This makes practically unfeasible, with present computational resources, the numerical study of the infrared behavior of the gluon propagator in four dimensions and at large values of β, and explains why a decreasing gluon propagator has been observed only in the strong-coupling regime for the four-dimensional case. [7,9]

In the three-dimensional case we get good scaling for values of β greater than 3.4, especially in the region of large momenta, where finite-size effects are negligible. We also see that the gluon propagator is decreasing for momenta $p \lesssim p_{dec}$, and that the value of p_{dec} (in physical units) is practically β-independent. From our data we obtain $p_{dec} \approx 350\,\mathrm{MeV}$. For the same set of data we also observe that the gluon propagator is less singular than p^{-1} in the infrared limit, in agreement with Zwanziger's prediction. We notice that the *turnover* momentum $p_{to} \approx 700\,\mathrm{MeV}$ is in good agreement with the result obtained recently in four dimensions for the $SU(3)$ group. [10] Finally, the value $D(0)$ of the gluon propagator at zero momentum decreases monotonically as the lattice volume increases (see for example the case $\beta = 5.0$ in the three-dimensional case [8]). This suggests a finite value for $D(0)$ in the infinite-volume limit, but it is not clear whether this value would be zero or a strictly positive constant. Therefore, the possibility of a zero value for $D(0)$ in the infinite-volume limit is not ruled out.

3.1 Conclusions

The prediction [4,6] of a gluon propagator decreasing for momenta $p(k) \lesssim p_{dec}$ is clearly verified numerically for several values of the coupling β. In the three-

dimensional case these values range from the strong-coupling regime to the scaling region. Also, our data in the strong-coupling regime for the three-dimensional case are in qualitative agreement with the results obtained in four dimensions. [7] This strongly suggests to us that a similar analogy will hold — in the limit of large lattice volumes — for couplings β in the scaling region, leading to an infrared-suppressed gluon propagator also in the four-dimensional case.

4 Discretization effects

The definition of the lattice gluon field given in Eq. (5) is only one of the possible lattice discretizations of $A_\mu(x)$, i.e., we can consider several possible definitions, leading to discretization errors of different orders. For example, we can write [11]

$$A^{(1)}_\mu(x) \equiv \frac{U_\mu(x) \,-\, U^\dagger_\mu(x)}{2\,i} \tag{8}$$

$$A^{(2)}_\mu(x) \equiv \frac{[U_\mu(x)]^2 \,-\, [U^\dagger_\mu(x)]^2}{4\,i} \tag{9}$$

$$A^{(3)}_\mu(x) \equiv \frac{[U_\mu(x)]^4 \,-\, [U^\dagger_\mu(x)]^4}{8\,i} \quad . \tag{10}$$

If we set

$$U_\mu(x) \equiv \exp[iag_0\,\vec{\sigma}\cdot\vec{\mathcal{A}}(x)]\;, \tag{11}$$

we obtain that $A^{(1)}_\mu(x)$, $A^{(2)}_\mu(x)$ and $A^{(3)}_\mu(x)$ are equal to $ag_0\,\vec{\sigma}\cdot\vec{\mathcal{A}}(x)$ plus terms of order $a^3\,g_0^3$. We can also consider

$$A^{(4)}_\mu(x) \equiv \Big[\,4\,A^{(1)}_\mu(x) \,-\, A^{(2)}_\mu(x)\Big]\;/\,3 \tag{12}$$

$$A^{(5)}_\mu(x) \equiv \Big[\,16\,A^{(1)}_\mu(x) \,-\, A^{(3)}_\mu(x)\Big]\;/\,15 \tag{13}$$

$$A^{(6)}_\mu(x) \equiv \Big[\,4\,A^{(2)}_\mu(x) \,-\, A^{(3)}_\mu(x)\Big]\;/\,3 \tag{14}$$

and

$$A^{(7)}_\mu(x) \;\equiv\; \Big[\,64\,A^{(1)}_\mu(x) \,-\, 20\,A^{(2)}_\mu(x) \,+\, A^{(3)}_\mu(x)\Big]\;/\,45\;. \tag{15}$$

It is easy to check that $A^{(4)}_\mu(x)$, $A^{(5)}_\mu(x)$ and $A^{(6)}_\mu(x)$ are equal to $ag_0\,\vec{\sigma}\cdot\vec{\mathcal{A}}(x)$ plus terms of order $a^5\,g_0^5$, and that $A^{(7)}_\mu(x) = ag_0\,\vec{\sigma}\cdot\vec{\mathcal{A}}(x)$ plus terms of order $a^7\,g_0^7$.

Table 1: The parameter Z and the χ^2/d.o.f. for the best fit to the ultraviolet behavior $Zg_0^2/\left[4\,p^2(k)\right]$. We consider different lattice discretizations of the gluon propagator, and indicate with $D^{(i)}_{TI}(k)$ the $D^{(i)}(k)$ propagator evaluated using tadpole-improved links $\widetilde{U}_\mu(x)$.

Type of propagator	$\beta = 2.7$		$\beta = 10$	
	Z	χ^2/d.o.f	Z	χ^2/d.o.f
$D^{(1)}(k)$	1.2965	1.36	1.0261	0.121
$D^{(1)}_{TI}(k)$	1.6052	1.36	1.0904	0.121
$D^{(2)}(k)$	0.9141	1.25	0.9295	0.133
$D^{(2)}_{TI}(k)$	1.4012	1.25	1.0496	0.133

The definitions $A^{(1)}$ and $A^{(2)}$ were recently considered by Giusti *et al.* [12] They find that the corresponding gluon propagators are equal modulo a constant factor. We have performed a similar study [11] at several values of β and for lattice volumes $V = 8^4$ and 12^4, evaluating $D^{(i)}(k)$ using the different definitions of the gluon field $A^{(i)}_\mu$ given above. We obtain, in all cases, that the seven propagators $D^{(i)}(k)$ are equal modulo a constant factor. We notice that this proportionality constant between different discretizations of the gluon propagator may be explained as a tadpole renormalization. In fact, let us consider the tadpole-improved link $\widetilde{U}_\mu(x) \equiv U_\mu(x)/u_0$, where u_0 is the mean link in Landau gauge. Then $D^{(1)}(k)$ gets multiplied by a factor u_0^{-2}, and $D^{(2)}(k)$ by u_0^{-4}. For example, for $V = 12^4$ and $\beta = 2.2$ we have $u_0^2 = 0.68428(8)$. In this case, using tadpole-improved operators, the discrepancy $D^{(1)}(k)/D^{(2)}(k)$ is reduced from 1.833(4) to 1.254(3). Similarly, at $\beta = 2.3$ [and 2.7], we obtain $u_0^2 = 0.7222(1)$ [respectively $u_0^2 = 0.8077(3)$] and the discrepancy $D^{(1)}(k)/D^{(2)}(k)$ is reduced from 1.701(4) [respectively 1.425(3)] to 1.228(3) [respectively 1.151(2)]. As expected, the discrepancy between different discretizations decreases as β increases.

It is also interesting to compare the data obtained for the gluon propagator at large momenta with the ultraviolet behavior $g_0^2/\left[4\,p^2(k)\right]$, predicted by perturbation theory at zeroth order. To this end, we consider different discretizations of the gluon propagator and we fit the data corresponding to $p^2(k) \geq 3$ using the function $Zg_0^2/\left[4\,p^2(k)\right]$. In Tab. 1 we report the results obtained for the lattice volume $V = 12^4$ at $\beta = 2.7$ and $\beta = 10$. Again, the discrepancy between $D^{(1)}(k)$ and $D^{(2)}(k)$ is reduced when tadpole-improved operators $D^{(i)}_{TI}(k)$ are used. Also, as expected, the value of Z gets closer to 1 as β increases.

Finally, let us notice that, in finite-temperature QCD, the long-distance behavior of the gluon propagator is directly related to the electric and magnetic screening lengths, and that these screening masses are invariant under rescaling of the propagators by a constant factor. In particular, $D_1^{(i)}(k) + D_2^{(i)}(k)$ [respectively $D_4^{(i)}(k)$] is related to the gluon propagator used by Karsch *et al.*[13] for the evaluation of the magnetic [respectively electric] screening mass. It has been checked[14] that, also at finite temperature, different discretizations for the gluon propagator are equal modulo a constant value, i.e., the screening masses are independent of the discretization $D^{(i)}(k)$.

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